Lecture 1: Lifting Brauer Characters and Generalized Class Functions

William Chuang, Based on Prof. Lux' Lectures and Navarro's Book

March 27, 2025

1. Motivation: From Ordinary to Modular Representation Theory

Let G be a finite group and p a prime dividing $|G|$. While ordinary character theory studies representations over $\mathbb{C}$, modular representation theory considers representations over a field k of characteristic p. The irreducible characters in this setting are called *Brauer characters*, denoted $\phi \in \mathrm{IBr}(G)$.

2. p-Regular Elements and Domains of Brauer Characters

Definition 1 (p-regular element). *An element $g \in G$ is called* p-regular *if the order of g is not divisible by p.*

Let:

- $G_{p'}$: the set of p-regular elements of G,
- $\mathrm{Cl}_{p'}(G)$: the set of p-regular conjugacy classes.

Brauer characters $\phi \in \mathrm{IBr}(G)$ are defined only on $G_{p'}$, and vanish on p-singular elements.

3. Generalized Characters and Class Functions

Let:

- $\mathbb{Z}\,\mathrm{Irr}(G)$: the group of generalized ordinary characters (i.e., integer linear combinations of $\mathrm{Irr}(G)$),
- $\mathbb{Z}\,\mathrm{IBr}(G)$: the group of generalized Brauer characters.

Given $\theta \in \mathbb{Z}\,\mathrm{IBr}(G)$, we aim to lift θ to a class function on all of G, ideally lying in $\mathbb{Z}\,\mathrm{Irr}(G)$.

4. Two Canonical Lifts of a Brauer Character

Let θ be a class function on $G_{p'}$. We define two lifts:

(1) Tilde Lift (Green's Lift)

Let $|G|_p$ denote the highest power of p dividing $|G|$. Define:

$$\tilde{\theta}(x) = \begin{cases} |G|_p \cdot \theta(x) & \text{if } x \text{ is } p\text{-regular}, \\ 0 & \text{otherwise.} \end{cases}$$

(2) Hat Lift

Let $x = x_p x_{p'}$ be the Jordan decomposition of $x \in G$, where x_p is the p-part and $x_{p'}$ the p'-part. Define:

$$\hat{\theta}(x) = \theta(x_{p'}).$$

5. Lemma (Navarro, Lemma 2.15)

Lemma 1 (Green's Lift Preserves Generalized Characters)**.** *Let $\theta \in \mathbb{Z}\,\mathrm{Irr}(G)$ or $\theta \in \mathbb{Z}\,\mathrm{IBr}(G)$. Then both $\tilde{\theta}$ and $\hat{\theta}$ lie in $\mathbb{Z}\,\mathrm{Irr}(G)$.*

This lemma is proven using a characterization theorem of Brauer (see Navarro, Chapter 2).

6. Brauer's Criterion for Generalized Characters

Theorem 1 (Brauer's Characterization Theorem)**.** *A class function θ on G lies in $\mathbb{Z}\,\mathrm{Irr}(G)$ if and only if for every p-elementary subgroup $E = P \times Q \leq G$, where P is a p-group and Q is a p'-group, the restriction $\theta|_E \in \mathbb{Z}\,\mathrm{Irr}(E)$.*

To verify that $\tilde{\theta}, \hat{\theta} \in \mathbb{Z}\,\mathrm{Irr}(G)$, it suffices to show their restriction to $P \times Q$ lies in $\mathbb{Z}\,\mathrm{Irr}(P \times Q)$.

7. Example: Regular Character of a p-Group

Example 1. *Let P be a nontrivial p-group. Then $\mathrm{IBr}(P) = \{\phi_{\mathrm{triv}}\}$, and the corresponding projective character is the regular character:*

$$\Phi(x) = \begin{cases} |P| & x = 1, \\ 0 & x \neq 1. \end{cases}$$

This is an example of a lift via $\tilde{\theta}$, concentrated on the identity.

8. Summary

- Brauer characters are defined only on p-regular elements.
- The tilde and hat lifts extend Brauer characters to all of G.
- Lemma 2.15 (Navarro) shows both extensions are generalized characters.
- Brauer's criterion gives a test for when a class function lies in $\mathbb{Z}\,\mathrm{Irr}(G)$, based on restrictions to p-elementary subgroups.

Next Lecture

In Lecture 2, we will:

- Introduce the decomposition matrix and its entries $d_{\chi\phi}$,
- Define projective indecomposable characters and their duality with $\mathrm{IBr}(G)$,
- Study orthogonality on p-regular classes,
- Examine the inner product matrix and dual basis relationships (see Navarro, Chapter 3).